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LETTER TO THE EDITOR

Temperley–Lieb operator formalism for Z_q symmetric models and solvable submanifolds

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Abstract. We give the operator formalism for the transfer matrix in Z_q symmetric models. We give the structure and irreducible representations of the associated operator algebra. We show that the Potts model limit gives a subalgebra which is the unitarisable quotient of the Temperley-Lieb algebra, and give the inclusion of this subalgebra. We discuss other subalgebras in the context of a conformal field theory limit of the Z_q symmetric models.

It has long been known how to write the n-site layer transfer matrix (TM) for the square lattice q-state Potts model in terms of representations of the Temperley-Lieb algebra $T_{2n-1}(q)$ with $2n-1$ operators $\{U_i\}$ obeying relations

$$\begin{aligned} &U_iU_i=\sqrt{q}\,U_i \\ &U_iU_{i\pm 1}U_i=U_i \\ &[U_i, U_{i+j}]=0 \qquad j>1 \end{aligned} \tag{1}$$

(Temperley and Lieb 1971 (hereafter referred to as TL), Baxter 1982). The Potts model may be regarded as a special case of the general Z_q symmetric model with Hamiltonian (or action in the field theory formalism)

$$H_{\{\beta_r\}}=\beta \sum_{\substack{\text{lattice}\\ \text{links } ij}} \chi_{\{\beta_r\}}(p_i-p_j) \tag{2}$$

where

$$\chi_{\{\beta_r\}}(p)=\sum_{r=1,[q/2]} \beta_r \cos[2\pi rp/q] \tag{3}$$

and where the site variables $p_i \in \{1, \ldots, q\}$. Strictly speaking, in this form we have described the non-chiral version of the general model. The constants $\{\beta_r\}$ determine the interactions for a given specific model, with $\beta_r=1/[q/2]$ for the Potts model, which has $\chi(p)=\delta_{p,0}$ (Baxter 1982).

An operator formalism is an invaluable tool in the analysis of equilibrium statistical mechanical models, and much play has been made of the TL formalism for the Potts model (see Kuniba *et al* (1986, hereafter referred to as KAW), for instance). In the present letter we will give the operator formalism for the Z_q model. We will explain why it is *the* formalism, describe the associated operator algebra and put this general work in the context of progress with the TL algebra.

The m-site TM for the Z_q model with zero boundary conditions at the layer edges (see Baxter 1982) is

$$T = \left(\prod_{i=1}^{m} M_{2i-1}\right)\left(\prod_{i=1}^{m-1} M_{2i}\right) \tag{4}$$

where M_i is the TM for a single bond, given by

$$M_i = \sum_{j=0}^{q-1} \alpha_{ij}[(S_i)^j] \tag{5}$$

where α_{ij} is a readily determined function of $\{\beta, \beta_r\}$ and the matrices S_i obey the relations

$$\begin{aligned} &S_i^q = 1 \\ &S_i S_{i+1} S_i^{-1} S_{i+1}^{-1} = w \qquad w = \exp(2\pi i/q) \\ &[S_i, S_{i+j}] = 0 \qquad j > 1. \end{aligned} \tag{6}$$

These matrices take the form

$$\begin{aligned} S_{2i} &= 1 \times 1 \times \ldots \times \begin{pmatrix} 1 & & & & \\ & w & & & \\ & & w^2 & & \\ & & & \ddots & \\ & & & & w^{q-1} \end{pmatrix} \times \begin{pmatrix} 1 & & & & \\ & w^{q-1} & & & \\ & & w^{q-2} & & \\ & & & \ddots & \\ & & & & w \end{pmatrix} \times \ldots \times 1 \\ S_{2i-1} &= 1 \times 1 \times \ldots \times \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{pmatrix} \times 1 \times \ldots \times 1 \end{aligned} \tag{7}$$

(a total $n = 2m-1$ operators) (see Martin 1986) where the unit matrices are q dimensional and the others appear at the ith position in the product. The basis of spin configurations is then as in Baxter (1982, p 334). It is easy to check that these objects produce the desired TM. For example,

$$\begin{aligned} &\alpha_{2i-1,j} = \exp(\beta\chi(j)) \qquad (= \alpha_{2i-1,q-j}) \\ &\alpha_{2i,j} = \sum_{r=0}^{[q/2]} \frac{w^{-jr}\exp(\beta\chi(r))}{q} \end{aligned}$$

for the homogeneous isotropic non-chiral models. The generalisation to chiral models is similarly straightforward.

Ordinary periodic boundary conditions may be obtained by including a factor M_0 in the TM, where $S_0 = S_{2m} = (\Pi_{i=1}^{m-1} S_{2i})^{-1}$. Seamed periodic boundary conditions (as, for example, in Baxter *et al* 1976, Pasquier 1988) require $S_0 = S_{2m} = w(\Pi_{i=1}^{2m-1} S_i^{-1})$. It is straightforward to check that either of these extensions completes a 'circle' of relations for the S operators. It is worth noting that, in each case, the extension to periodic boundaries is completed within the existing algebra.

To obtain the structure of the algebra defined by the above relations it is useful to also consider the representation associated with free boundary conditions (the first and last spin in each layer interacting only within the layer, i.e. free from interactions with adjacent layers) which are dual to the above boundary conditions (in the sense of Savit (1980))

$$S_{2i-1}=1\times 1\times\ldots\times\begin{pmatrix}1&&&&\\&w&&&\\&&w^2&&\\&&&\ddots&\\&&&&w^{q-1}\end{pmatrix}\times\begin{pmatrix}1&&&&\\&w^{q-1}&&&\\&&w^{q-2}&&\\&&&\ddots&\\&&&&w\end{pmatrix}\times\ldots\times 1$$

$$S_{2i}=1\times 1\times\ldots\times\begin{pmatrix}0&1&&&\\&0&1&&\\&&0&1&\\&&&\ddots&1\\1&&&&0\end{pmatrix}\times 1\times\ldots\times 1 \tag{8}$$

(a total $n=2m+1$ operators) and with self-dual boundary conditions (zero at one end, free at the other), e.g. with $m=2$, $q=4$,

$$S_1=\begin{pmatrix}1&&&\\&i&&\\&&-1&\\&&&-i\end{pmatrix}\times\begin{pmatrix}1&&&\\&-i&&\\&&-1&\\&&&i\end{pmatrix}\times 1$$

$$S_2=1\times\begin{pmatrix}0&1&&\\&0&1&\\&&0&1\\1&&&0\end{pmatrix}\times 1$$

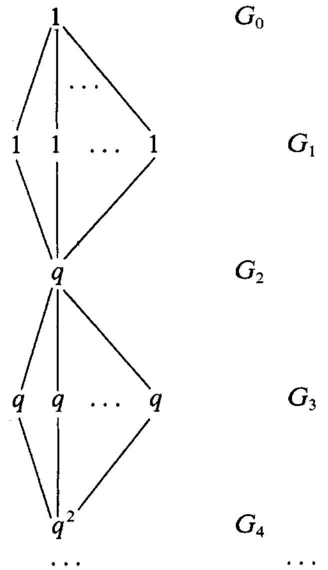
$$S_3=1\times\begin{pmatrix}1&&&\\&i&&\\&&-1&\\&&&-i\end{pmatrix}\times\begin{pmatrix}1&&&\\&-i&&\\&&-1&\\&&&i\end{pmatrix} \tag{9}$$

$$S_4=1\times 1\times\begin{pmatrix}0&1&&\\&0&1&\\&&0&1\\1&&&0\end{pmatrix}$$

($2m$ operators in general).

It is easy to see that the algebra $G_n(q)$ ($q\in\mathbb{Z}$) defined by n operators obeying the relations above is q^n dimensional and that the self-dual representation consists of q copies of a q^m-dimensional irreducible representation ($n=2m$). The $n=2m$ operator algebras are thus isomorphic to $M_{q^m}(\mathbb{C})$, the q^m-dimensional matrix algebras. Similarly the zero boundary condition representations for the $n=2m-1$ operator algebras contain exactly one copy of each of the q inequivalent irreducible representations of dimension q^{m-1}, i.e. $G_{2m-1}(q) \simeq qM_{q^{m-1}}(\mathbb{C})$. It is in this sense that the Z_q model and the $G_n(q)$ algebra are uniquely related. The irreducible representations may most easily be read off from the free boundary representation, by discarding the last (or first) factor in the cross product.

The Bratteli diagram (see, e.g., Jones 1983) for the inclusion of $G_{n-1}(q)$ in $G_n(q)$ is



and the centre of $G_{2m-1}(q)$ is generated by $S_1S_3 \ldots S_{2m-1}$. For $n=2m+1$ the matrix structure of the algebra is exhibited by the identities

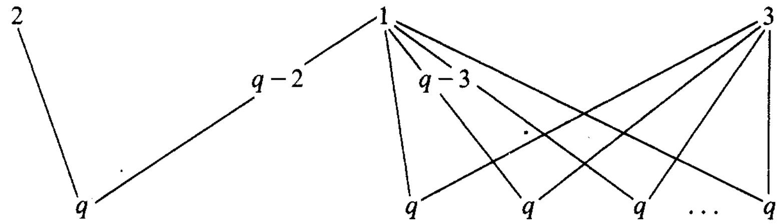
$$\frac{S_2^p U_1 S_2^{-p}}{\sqrt{q}}\left(\prod_{i=1}^{m}\frac{U_{2i+1}}{\sqrt{q}}\right)\left(\prod_{i=1}^{m} S_{2i}^{-\beta_i}\right)\left(\prod_{i=1}^{m} S_{2i}^{\gamma_i}\right)\frac{S_2^r U_1 S_2^{-r}}{\sqrt{q}}\left(\prod_{i=1}^{m}\frac{U_{2i+1}}{\sqrt{q}}\right)$$
$$=\delta_{p,r}\left(\prod_{i=1}^{m}\delta_{\beta_i,\gamma_i}\right)\frac{S_2^p U_1 S_2^{-p}}{\sqrt{q}}\left(\prod_{i=1}^{m}\frac{U_{2i+1}}{\sqrt{q}}\right) \tag{10}$$

$(p, r, \beta_i, \gamma_i \in 0, \ldots, k-1)$ where the operators $\{U_i\}$ are defined in (11) below. We may use these identities to reduce any calculation involving the Z_n TM to a purely operator algebraic one (cf Baxter (1982) for the Potts model case).

It is straightforward to check that the objects

$$U_i = (1/\sqrt{q})\left(\sum_{j=0}^{q-1} S_i^j\right) \tag{11}$$

obey the TL relations and that $\alpha_{ij}=$ constant $(j \neq 0)$ in (5) recovers the Potts model TM up to overall factors. We note from this and by reference to Martin (1988a) that the 'U subalgebra' of $G_n(q)$ generated by the objects $\{U_i\}$ above is precisely the unitarisable quotient of the TL algebra, which is given in Jones (1983) or Martin (1988a) and is, for example, $M_1(\mathbb{C})$ for $q=1$, the Clifford algebra for $q=2$, and the whole TL algebra for $q \geqslant 4$. The inclusion of the U subalgebra in $G_n(q)$ may readily be deduced from the above using Martin (1988a). For example with $n=2$, $q \geqslant 3$ we have



where the degeneracies are unity unless indicated otherwise. We read from this diagram that the Potts TM is isomorphic to one two-dimensional block; $(q-2)+(q-1)(q-3)=q^2-3q+1$ equivalent one-dimensional blocks and $(q-1)$ equivalent three-dimensional blocks, while the general model has one distinct q-dimensional block and $(q-1)$ equivalent q-dimensional blocks.

The interest in such subalgebras lies in the simplification (block diagonalisation) of the transfer matrix implied by inclusions such as the one above. A practical example of this may be found in Martin (1988b). In the Potts cases it is precisely this which enables the model to be solved at criticality (Baxter 1982) and with $q=2$ to be solved completely (Onsager 1944). The simplification of representations with $q \leqslant 4$ parallels the solution of conformal field theories (Belavin *et al* 1984) associated with the same models (Friedan *et al* 1984).

When $q>4$ the Potts models have first-order phase transitions and therefore no such limit. However, it has been widely conjectured (see, e.g., Huse 1984) that there are Z_q symmetric models (probably at the crossover between two- and three-phase models for $q \geqslant 5$) with field theory limits associated with the conformal series. It becomes an important problem, therefore, to identify the other subalgebras of $G_n(q)$. If q contains p as a factor then $G_n(q)$ has $T_n(p)$ as a subalgebra, but this is, in effect, well known, corresponding to the existence of Z_p-like representations of Z_q, where the generators are

$$\bar{U}_i \propto \sum_{j=0}^{[p]} S_i^{(q/p)j}.$$

What we really want is subalgebras of $G_n(q)$ $(q>4)$ which are commuting subalgebras of $T_n(4\cos^2(\pi/r))$ $(r(q) \in \mathbb{Z})$ corresponding to the commuting TM of Baxter (1982) (see also Huse 1984, Friedan *et al* 1984, Kuniba *et al* 1986). Work on this aspect is in progress. In particular, the operator which conjugates S_i to S_{i+1} (and S_{2m-1} to S_0 in G_{2m-1}, whose $2m$th power is central, cf the braid group (Birman 1974)) would be useful as the generator of translations.

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